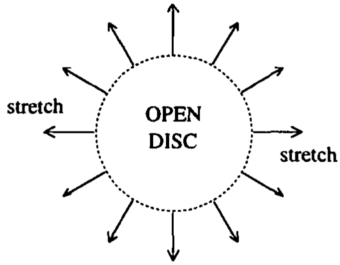
AN INTRODUCTION TO TOPOLOGY

Topology is a kind of abstraction from metrical geometry. The metrical geometry is the distance geometry of a space and gives rise to concepts such as length, angles, and curvature. Topology studies spaces with a much more general conception of "nearness" than that provided by the metric. Thus, although the metric geometry distinguishes spheres, cubes, and pyramids from one another due to their different metrical properties, topology classifies them together as instances of the same object. However, topology does mark a difference between spheres and doughnuts (or tori) since, loosely speaking, "holes" are topological properties. Explaining the difference between topological and metrical properties is therefore the natural first order of business when introducing the ideas of topology.

Let's begin in an intuitive way and think about two dimensional surfaces. The way we distinguish the topology of a surface from its metrical geometry is by stating that topology concerns those properties of a surface that are invariant under continuous (for now, elastic) transformations, while the metrical geometry is the distance geometry of the surface. As we saw, sameness of topology does not imply sameness of metrical geometry, for spheres and cubes are the same topological object. But what about the converse? To investigate this consider the infinite plane and the infinite cylinder. Topologically these are distinct because they cannot be continuously transformed into each other. Thus, consider an open disc (a disc without a boundary). It is topologically equivalent to the infinite plane because it can be continuously transformed into the infinite plane depicted (so being finite or infinite are not topological properties!).



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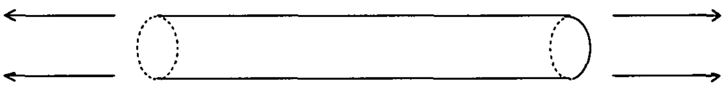
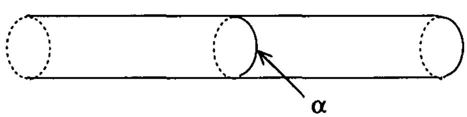
Next, form the punctured disc by removing one point from the open disc. By similar reasoning we see that the punctured disc is topologically equivalent (≈) to the punctured plane (the plane with one point removed). But it is also easy to see that the punctured disc is topologically equivalent to a finite open cylinder (a cylinder without boundaries). Just stick your finger in the puncture and pull it up (or out) to form the cylinder, which, as the arrows indicate, is topologically equivalent to an infinite cylinder.



Therefore, the punctured plane ≈ punctured disc ≈ infinite cylinder, and by the transitivity of "can be continuously transformed into", the punctured plane ≈ infinite cylinder. However, the plane cannot be continuously transformed into the punctured plane. By smoothly stretching and compressing the plane we simply cannot form a puncture, so the punctured plane ≉ infinite plane, and so, the infinite plane ≉ infinite cylinder.

More simply, let us say that a surface is *simply connected* iff every simple (i.e., non-intersecting) closed curve can be continuously contracted to a point. The plane is simply connected while the cylinder is not. As we see in the picture, α, which goes "around" the cylinder, can not be smoothly contracted to a point.



Being simply connected is a topological property, so once again we conclude the two surfaces are not topologically equivalent.

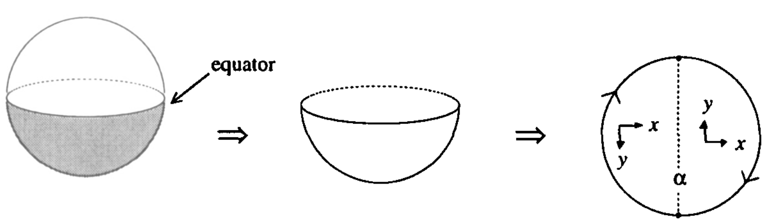
Metrically, however, the plane and the cylinder are equivalent in the following sense. Since a rectangular sheet can be rolled up into a cylinder without stretching or tearing it, a two dimensional being living in a cylinder cannot, by making distance measurements (measuring the lengths of curves), tell whether he is living on a cylinder or a plane, *if* he confines himself to a simply connected region. Of course, if he makes a trip all around the cylinder, he can tell he does not live in the plane. But then he is making use of a topological property, namely, that the cylinder is not simply connected.

We have seen that the cylinder and the plane are not topologically equivalent because one is simply connected and the other is not. This gives rise to the questions: are all (non-) simply connected topologies equivalent? The answer in each case is no, as it is easy to see. First, consider a sphere (S^2) and the plane. Both are simply connected yet they are obviously not topologically equivalent. For example, the punctured sphere—the sphere missing a point—is, as the reader should now be able to see, topologically equivalent to the infinite plane. Since the sphere is not topologically equivalent to the punctured sphere, it is not topologically equivalent to the plane. Interestingly, this shows that puncturing a simply connected surface does not necessarily make it non-simply connected.

Of course, we know intuitively that the sphere cannot be continuously transformed into the plane without tearing it. This is because the surface of the sphere closes around into itself in all directions, i.e., that it completely encloses a volume. But that is a characterization that depends on the two spheres embedding in three dimensional space. We would like to characterize the topology of a sphere in an intrinsic way, using only two dimensional notions. A two dimensional being confined to the surface, who knew nothing of three dimensional space, could still characterize the topology of his world using properties that make no reference to an embedding space. For example, he can discover that his universe is finite but unbounded. We have here used the metrical notion of finite. It turns out that topologically the crucial property is compactness, which we will soon study.

As in the case of simply connected surfaces, not all non-simply connected surfaces are topologically equivalent. Just compare the cylinder with the torus or doughnut. As in the previous case of the plane and the sphere, the cylinder and the torus differ in that the former is not compact but the latter is.

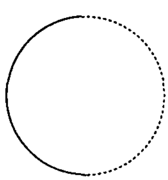
So far we have been thinking intuitively about surfaces by picturing them in three dimensional space. But there are surfaces (two dimensional "manifolds") that cannot be so embedded. That is, there are surfaces that are not topologically equivalent to any subset of three dimensional Euclidean space. An important example is the projective plane. We can obtain a model of the projective plane by identifying antipodal points on the surface of a sphere. The projective plane is represented by the sphere when we regard antipodal points *as the same point*. To obtain a simpler picture, note that if we throw away all the points above the equator, we still have all the points of the projective plane represented.



Flattening the remaining hemisphere (including the equator), the projective plane can be represented by a disc with opposite points on the boundary being identified. Like the sphere, the projective plane closes in on itself—it is compact (and we can see here that being compact is not in any simple way equivalent to enclosing a volume). But unlike the sphere it is not simply connected. A curve like α, for example, cannot be continuously contracted to a point. Further, unlike the plane, cylinder, sphere and torus, it is not orientable. A left-right orientation can not be consistently established at each point. For example, if "→" is right handed we see that it becomes left-handed "←" as we move through the "boundary" from one side to the other.

The projective plane is our first example that topology applies to more exotic two dimensional spaces than the usual surfaces in three dimensional Euclidean space. In order to be able to extend the notions of topology to even more exotic spaces, higher dimensions, and many other structures such as groups, we need to generalize and make precise, the notion of topological equivalence that we intuitively explained in terms of smoothly stretching and compressing surfaces. We need, in the first instance, to give a precise and general enough characterization of continuity.

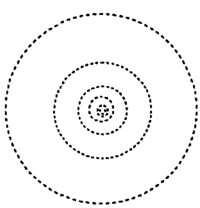
The key to this (and thus to topology) is the notion of an *open* set. In the plane and other two dimensional surfaces, the paradigm open sets are the interiors of discs (open discs). They form a *basis* for the set of open sets of the surface, i.e., any open set is a union of open sets from the basis. In order to ensure that the union of arbitrary opens sets is open, we take the whole space to be open, and it's convenient to take the empty set to be open as well. *Closed* sets are then defined to be the complements of the open sets, so the whole space and the empty set are to be both open and closed. While the paradigmatic open set is a disc minus its boundary, a disc with its boundary is a paradigmatic closed set. However, not all sets are either open or closed. While some are both, plenty are neither, e.g., a disc plus only part of its boundary.



Let's be more explicit about what we mean by the boundary of a set. Note that for a point p on the boundary of a disc, each open set containing p contains points in the disc and points not in the disc. Consequently, we say that for any set Y, the boundary $B(Y)$ of Y equals the set of points p such that any open set containing p contains points in Y and points in Y^c, where Y^c is the complement of Y. An equivalent way of putting this is to define the *interior* of a set Y, $I(Y)$, as the union of all the open sets contained in Y, and the *exterior* of Y to be the interior of Y^c. Then the boundary of Y are those points neither in $I(Y)$ nor $I(Y^c)$. It follows that a set is closed iff it contains its boundary (because its exterior is its complement).

Let's consider some examples. Let S be a surface with its standard topology and a discrete set of points. Intuitively, we know Y^c is open so Y must be closed. More precisely, the only boundary points of Y are the points of Y themselves, so Y must be closed, and in fact, $B(Y)=Y$. Let S equal an infinite line that we identify with the set of real numbers and let Y be the subset consisting of all the rational points. Any open set, e.g., the open interval, contains both rationals and irrationals, so $B(Y)=S$. Let S be the real line and Y the closed interval between 0 and 1, [0, 1]. The boundary of Y consists of the two points 0 and 1. But suppose S is the plane and Y is [0, 1] on the x-axis. Relative to the new S, Y is still closed (it contains all its boundary points), but $B(Y)=Y$.

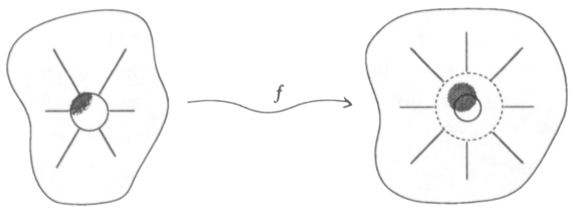
Returning to open sets, the important point is that the topology of a surface (or any set) is coded into the set of its open sets. For example, consider what happens when we puncture the plane. The puncture is contained in a nested series of open sets. If it is at the origin, then we can consider the series of open sets that are the interiors of circles of radius $1/n$, $n \geq 1$.



Before the puncture the intersection of these open sets is not empty, it consists of just the puncture point. But after the puncture, their intersection is empty (note that after the puncture they are still all open).

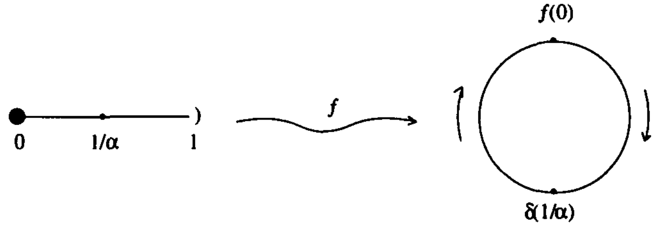
Further, we can now define continuity. Since a continuous transformation of a surface is really a map or function from one surface to another, what we really need to define is the notion of a continuous function. First, then, we define a function f, from one set M *into* another set N, $f: M \rightarrow N$, to be a rule that associates with each element $x \in M$ a unique element $y=f(x) \in N$. If, in addition, for each $y \in N$, there exists an x such that $f(x)=y$, f is *onto*.

Now consider two surfaces M, N and a map $f: M \rightarrow N$ that works as pictured below. The circles are centered on the origin of the coordinates in both, the circles of radius $r \leq n$ are mapped onto the corresponding circles in N, the circles of radius $p > n$ in M are mapped to circles of radius $p + 1$ in N. Therefore, in mapping M to N, f tears M since f maps N into two disjoint pieces separated by a region (the annulus) of N that is not in the range of f. Because of this we can form an open set of N that contains points both in and out of the range of f, as pictured, whose inverse image (the set of points of M mapped into that set) is not an open set since it contains part of its boundary. The inverse image does not preserve the open set structure. If in mapping M to N no tear was produced, then we can see that the inverse image of an open set in N would be open in M and the map would be continuous. It turns out, then, that this is the crucial property of a continuous function, and our general definition of a continuous function f: $M \rightarrow N$ is: f is *continuous* iff for any open set $x \subseteq N$, $f^{-1}(x)$ is open in M, where $f^{-1}(x)$ is the set of points in M that f maps into x.



However, note that f being continuous does not fully capture our idea of topological equivalence. Because the plane is topologically equivalent to an open disc, it can be continuously mapped into an open disc that is part of the sphere. Clearly, that f is continuous and into is not enough for topological equivalence. f must be onto as well, and it must be 1–1. The plane can be mapped continuously into an open interval of the real line, but such a map cannot be 1–1. In other words, dimensionality is a topological property (or invariant).

One more condition on f is still needed for topological equivalence, namely, that the inverse of f (that exists because f is 1–1) is also continuous. Here is a simple example showing that f being continuous, 1–1 and onto is not enough to imply f^{-1} is continuous. Let M be the set [0, 1) that is the half open, half closed interval of the real line between 0 and 1 (the set of points x, $0 \leq x < 1$). Let N be a circle and f a function mapping M continuously onto N as pictured, in the clockwise direction. As defined the map is 1–1. Intuitively, however, M and N are not topologically equivalent and this is reflected in the fact that f^{-1} is not continuous. As the reader has probably guessed, the problem is $f(0)$. Consider the inverse image of f^{-1} on [0,1/2). Since f is 1–1, that is just $f([0,1/2))$ which is $[f(0), f(1/2))$ which is not an open set in the circle S^2. But [0,1/2) is an open set of M since [0,1) is the whole space M. True, considered as part of the infinite line, [0, 1) is not open, and neither is [0, 1/2), but in our case the complement of [0, 1/2) is [1/2,1) which is closed (it contains its boundary {1/2}).



Our definition of topological equivalence for surfaces (and related spaces) is then: Two surfaces M and N are *topologically equivalent or homeomorphic* iff there exists a 1–1 onto function f: $M \rightarrow N$ such that both f and its inverse f^{-1} are continuous. Such a map, called a homeomorphism, establishes a 1–1 correspondence between the open sets of M and N. It is in this sense that all the information about the topology of a surface is contained in the open sets.

We are now ready to be fully general. Given a set S, we call a set of subsets, T, of S, a *topology* for S iff the following conditions are satisfied:

(1) T is closed under arbitrary unions
(2) T is closed under finite intersections
(3) $S \in T$ and $\phi \in T$.

By definition T is the set of open sets of S *relative* to the topology T. The conditions (1) and (2) are abstracted from the behavior of the familiar open sets of Euclidean space, surfaces, etc. In particular, note that in (2) we are limited to finite, not arbitrary, intersections. For example, in the real line consider the open sets (relative to the standard topology) of the form $(-1/n, +1/n)$, for $n \geq 1$. The only point they have in common is 0, so

$$\cap\,(-1/n, +1/n) = \{0\}$$

and $\{0\}$ is a closed set.

Note also the emphasis on "relative". In general a set does not have a unique topology. Any subset of S that satisfies (1) through (3) specifies a topology for S. For example, given any set there are two limiting cases, the discrete topology T_d, which consists of all the subsets of S,

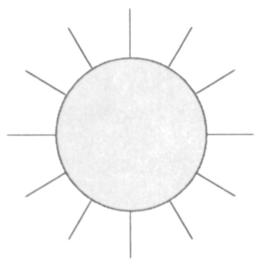
$$Y \in T_d \text{ iff } Y \subseteq S,$$

and the trivial or indiscrete topology T_i that consists of just S and ϕ,

$$Y \in T_i \text{ iff } Y = S \text{ or } Y = \phi.$$

While not the topology we are used to, these are quite useful for getting a feel for the definitions and testing general assertions. The important point is that when we say a set has a certain topological property, such as being compact or being topologically equivalent to another set, it is always relative to a given topology. In fact, it's common practice to call the pair $\{S,T\}$ a topological space, so that the claim that two topological spaces are homeomorphic or that one is compact, is automatically relative to a given topology T.

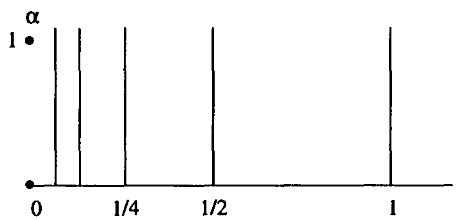
A nice example of this relativity occurs in the topology of the real line. It concerns the important topological property of being connected. We say that a topological space $\{S,T\}$ is *connected* iff there *do not* exist disjoint, non-empty sets X, $Y \in T$ such that $X \cup Y = S$. The intuitive idea goes like this. Consider the plane with its usual topology. If we try to divide it into two non-empty disjoint sets they can't both be open. For example, we can divide it into a closed disc and its complement (its exterior) which is open



But if we remove the disc's boundary, then we do get two disjoint open sets, but their union does not equal the plane. A connected set fits together as a single piece and intuitively this requires open sets fitting together with their complements, closed sets. Two disjoint non-empty open sets can not, intuitively, fit together.

But note, this notion of connectedness is not the strongest one possible. Indeed, maybe the most intuitive notion of being connected is that any two points in the topological space can be joined by a continuous path. Such a space is said to be *arcwise connected*. To make this precise, we need to define the notion of a continuous path between two points a and b. The definition is pretty much what one would expect: A *continuous path from a to b in S is a continuous function* f: $[\alpha,\beta] \to S$, from a closed interval of the real line into S, such that at $f(\alpha)=a$, $f(\beta)=b$. That is, the continuous function f transfers the continuity of the interval to a subset of S. In fact, you might think that the path in S is really the image of $[\alpha,\beta]$ under f. But the definition is customary. Anyway, the point is that while

path (arcwise) connectedness $\Rightarrow$ connectedness, the converse does not hold in an arbitrary topological space. It does hold in the usual spaces we are familiar with, but there are exotic counterexamples. One of these is the so-called deleted comb space. This space is a subset of the plane consisting of the x-axis from [0,1], all of the lines of constant $x = 1/n$ from $y = 0$ to $y = 1$, and the point $\alpha = (x = 0, y = 1)$, as depicted below. Without α it is obviously both connected and path connected. Including α it is still connected but there is no continuous path from α to any other points in the space.

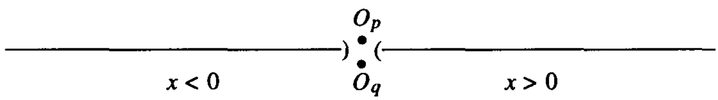


We can now return to the relativity we mentioned earlier. We consider the real line with the standard topology T_s composed of all the open intervals (a,b) and their arbitrary unions and with the *lower limit topology* T_l. T_l is composed of all the (according to T_s) half open intervals $[a,b)$ and their arbitrary unions. Our example of relativity is then that the real line in the standard topology is connected in T_s but not T_l. This follows because, for instance, $A = \cup[a,b)$ and $B = \cup[c,a)$ are both open (in T_l), disjoint, non-empty and $A \cup B = R$. Because connectedness is a topological property, the real line with the standard topology can not be homeomorphic to the real line with the lower limit topology. Thus, not even the identity map, i, from R_s to R_l can be a homeomorphism. It's 1–1, onto and continuous but i^{-1} is not continuous, i maps non-open sets of R_s into open sets of R_l. Remember that $[a,b)$ is not open in R_s, and note that (a,b) is open in R_l!

Next, let's finally explain compactness. The definition is a bit more complicated than the others and requires a little patience on the part of the student of topology. The crucial idea is that of the open cover. A set of open sets X_α is an *open cover* for a set $Y \subseteq S$ iff $Y \subseteq \cup X_\alpha$, i.e., iff every part of Y is included in the union of the X_α. Then a subset $Y \subseteq S$ of a topological space $\{S, T\}$ is *compact* iff every open cover of Y has a finite subcover of Y.

Let's try to get a feeling for this definition. It turns out that, in its usual topology, closed intervals $[a,b]$ of the real line are compact, but open intervals (a,b) and the half open intervals $(a,b]$, $[a,b)$ are not compact. For example, the union of the sets $(1/n,a]$, $n \geq 1$ cover $(0,a]$ $(a > 0)$, but no finite subcollection of the $(1/n,a]$ cover $(0,a]$. However, not all closed sets in R_s are compact. Thus the discrete subset $\{0, 1, 2, 3, 4, ...\}$ is closed but not compact. The open cover $(n - 1/4, n + 1/4)$, for all $n \geq 0$, has no finite subcover that covers the set since $(n - 1/4, n + 1/4)$ covers only n. Therefore, all the sets of the cover are needed. Nevertheless, one can easily show that every closed subset Y of a compact space S is compact. Since $S - Y$ (Y^c) is open, every open cover of Y with Y^c added to it becomes an open cover of S. Since S is compact the open cover X_α plus Y has a finite subcover of S and thus of Y. So Y is compact.

What about the converse question, are all compact sets closed? Here we can give a general answer using the topological property of being Hausdorff. A topological space S is *Hausdorff* iff for any two distinct points $x, y \in S$, there exist disjoint open sets O and O' such that $x \in O$, $y \in O'$. In other words, in a Hausdorff space any two distinct points can be separated from each other by disjoint open sets. For a picture, consider the real line where we double the point O.



The open sets are all sets $(a, 0)$, $(0, a)$, $(a < 0, O_p, b > 0)$, $(a < 0, O_q, b > 0)$ and their unions. By inspection we see that any open set containing O_p intersects one containing O_q.

Note that in the plane, say, single points are closed sets since they are their own boundary (their complement is open). But in a non-Hausdorff space, a single point can fail to be closed. A simple example is the set $\{a,b,c\}$ with the topology

$$T = \{\{a\},\{a,c\},\{a,b\},\{a,b,c\},\{\phi\}\}.$$

The set $\{a\}$ is not closed since its complement in S, $\{b,c\}$, is not in T so it is not open. And $\{a,b,c\}$ is not Hausdorff since b and c can not be separated by disjoint subsets. However, if one makes S Hausdorff by adding $\{b\}$ and $\{c\}$ to T, we must also add $\{b,c\}$ so that T is closed under arbitrary unions. This would make $\{a\}$ closed (as well as open) since its complement is now in T.

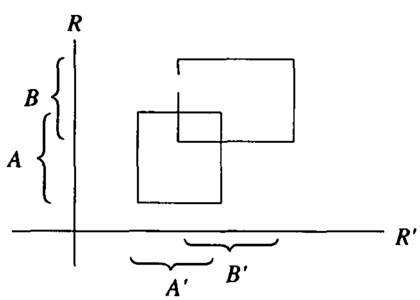
Returning to the question of whether every compact set is closed, we can answer that this is true in a Hausdorff space. To get a feel for this answer, let's look at a non-Hausdorff space with non-closed compact sets. The space is the real line with the finite complement topology,

$$T_f = \{R, \phi, \text{all complements of finite subsets}\}.$$

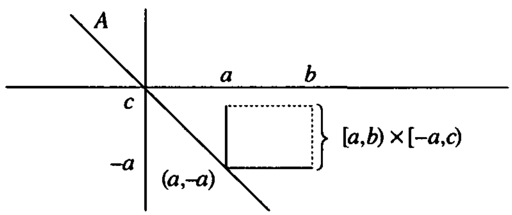
In this topology R (really R_f) is not Hausdorff, for given any two real numbers r_1, r_2, and open sets O_1, O_2, $r_1 \in O_1$, $r_2 \in O_2$, $O_1 \cap O_2$ will be non-empty since each lacks at most a finite number of real numbers, i.e., they will have an infinite number of points in common. Every subset of R_f is compact, and so, the non-closed open sets are too. Any such open set differs from all the others by only a finite number of points. So for the open cover of O we only need a finite number of covering sets to cover O.

Earlier we said that surfaces like the sphere and torus that close around themselves are compact and without boundary. Compare a sphere with an open disc in the plane. If one removes one point from the sphere it is homeomorphic to the open disc and thus not compact. If one adds its boundary circle to the open disc the result is also compact but it has no boundary. Similarly, if one adds the point back to the punctured sphere the result is also compact but without boundary. That point is something like a boundary in the sense that returning it results in a compact set, but when in place, the set still has no boundary at all!

Suppose we have two topological spaces. Can they be combined into a new topological space? Yes, and the most important way of doing this is by forming their topological product. Let S and S' be two sets with topologies T and T', respectively. We have two questions to answer. What is the product space, written $S \times S'$, and what is the product topology? The first is easy. $S \times S'$ is simply the Cartesian product of S and S'. That is, $S \times S'$ is the set of pairs (x,x'), $x \in S$, $x' \in S'$. For the second we have to be careful. It is tempting to say that the product topology $T \times T'$ consists of all the products of sets in T with those of T', i.e., that $T \times T'$ consists of all the product sets $A \times A'$, $A \in T$, $A' \in T'$. But this is not quite right. For let the real line cross itself, $R \times R$. Intuitively, this is the plane, each point of which is a pair (x,y), $x \in R$, $y \in R'$. According to our suggestion $A \times A'$ and $B \times B'$ would be in the product topology, but their union $A \times A' \cup B \times B'$ would not be, since it is not equivalent to $(A \cup B) \times (A' \cup B')$. As we can see from the picture above, $T \times T'$ would not be closed under unions. To remedy this we do the obvious and define the product topology $T \times T'$ to be the set of all unions of the sets of the form $A \times A'$, where $A \in T$, $A' \in T'$, i.e., we take the set of all products $A \times A'$ to be a basis for the topology of $S \times S'$.



Many familiar topological spaces can be viewed as topological products. The plane is $R \times R = R^2$, or more generally, n-dimensional Euclidean space has the topology R^n. As the reader would expect, the torus is $S^1 \times S^1$ and the cylinder is $R \times S^1$. And importantly, many topological properties such as compactness, connectedness, simple connectedness, being Hausdorff, etc., are preserved by taking the product $S \times S'$, if both S and S' have the property of interest. Nevertheless, not all properties are preserved. A nice example uses our old friend, R_l. R_l, while not compact, has the property that any infinite cover has a countable (finite or denumerably infinite) subcover, but this is not true of $R_l \times R_l$. To see this, consider the line A with coordinates $x = -y$ and open sets of the form $[a,b) \times [-a,c)$. Such an open set will intersect A at one point, $(a, -a)$, as pictured.



Therefore, $R_l \times R_l - A$ plus all the sets of the form $[a,b) \times [-a,c)$ constitute an open cover for $R_l \times R_l$. But since A has an uncountable number of points, there is no countable subcover. (Note: one can show that $R_l \times R_l - A$ is open in $R_l \times R_l$).

One more point about products. When we form infinite products, such as R^w, it's standard to use a different definition of the product topology.

Finally, let's briefly discuss the relation between topology and distance. We have already said that the distance between two points, or the distance geometry, is not a topological invariant (property). Nonetheless, there is a close and important connection between the two. To understand it we must first define the notion of a metric on a set S. It is a function d: $S \times S \rightarrow R$, from pairs of points in S into the real numbers that satisfies the following conditions. For $x,y \in S$

(1) $d(x,y) = d(y,x)$
(2) $d(x,y) \geq 0$ ($d(x,y) = 0$ iff $x = y$)
(3) $d(x,y) + d(y,z) \geq d(x,z)$.

d is a generalization of the distance between points in the Euclidean plane, where 1 and 2 are obvious and 3 is the so-called "triangle inequality", that the distance along two sides of a triangle must be greater than or equal to the remaining side. Therefore, we will call $d(x,y)$ the distance between x and y. Further, for any point p of a set S and given metric d on S, we call the set of points whose distance from p is less than ε, the ε ball, E(p,ε) centered on p. As one can demonstrate, the set of ε balls of S form the basis for a topology for S. This topology is called the topology *induced* by d.

Any set admits a metric. For example, take the trivial metric d such that for any x,y, $x \neq y$, $S(x,y) = 1$. Note that for any point p, $E(p,1/2)$ contains only P so a subset containing just one element is open in the induced topology. It follows immediately that the trivial metric induces the discrete topology. But while any set admits a metric (really, many metrics), not every topological space admits a metric, that is, a metric that induces the given topology. A simple example is any finite set S with more than one element with the indiscrete topology. Given any metric d on the set, there exists a number δ such that for *any* $x,y \in S$, $d(x,y) < \delta$. In other words, because there are only a finite number of points, points cannot be arbitrarily close to each other. But then, for any point p, the ball $E(p,\delta)$ contains only p, and p is open in the induced topology. And so the topology induced by d is not the indiscrete topology, since the only open sets in that case are S and ϕ. Interestingly, R_l is also not metrizable. More generally, there are beautiful theorems that state necessary and sufficient conditions for a topological space to be metrizable.

Earlier we mentioned that the notions of finite and infinite are not topological notions. The infinite plane and the open disc are topologically equivalent, for instance. In terms of a metric, what is going on is the following. Let us say that a set A of S is bounded in the metric d iff there is a number M such that $d(x,y) \leq M$ for every $x,y \in S$. Then we can show that the metric d_b on S, defined by d_b = minimum ($d(x,y)$, 1) induces the same topology as d. According to d_b, every set of S is bounded, even if according to d there are unbounded sets.

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FURTHER READING

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