Insights into Quantum Time Reversal from the Classical Schrödinger Equation

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1 Introduction

Standard non-relativistic quantum mechanics is widely presumed to be time reversal invariant. Yet the justification for this claim has always been a bit mysterious. The literature produces many distinct justifications. Most are not very convincing. And the symmetry often feels imposed rather than discovered, as when author Paul Roman writes that "we should like, even if only by some artifice, to achieve full covariance [time reversal invariance]" (Roman 1950, 266). Add to this impoverished motivation the fact that the time reversal operator is the only anti-unitary operator we use and it becomes natural to find the the conventional wisdom a bit suspicious – or at the least, worth greater elucidation.

In what follows I provide a new insight into this issue. First, I show that the *exact same puzzle* that arises in quantum mechanics arises also in classical physics. I do this by introducing the "quantum-looking" representation of classical theory of Schiller 1962 and Rosen 1964. That this puzzle can arise classically suggests that the problem is due to the representation and not the theory. And just as classical physics is time reversal invariant despite this puzzle, this observation provides hope that the conventional wisdom about quantum mechanics is right and that it is genuinely and unambiguously time reversal invariant. Second, and more controversially, I use the counterpart of the classical resolution of the puzzle to vindicate a definitional link between spatial variation of the phase and temporal variation. I show how Bohmian mechanics (and some other interpretations) provides this crucial step. With Bohmian mechanics playing the same role Newtonian mechanics does in the solution to the classical puzzle, I provide a satisfying way out of the quantum puzzle.

Put loosely, I argue that the puzzle of quantum time reversal arises because we have been given only "half" of the theory. By putting classical mechanics in a representation that reproduces the same questions, we can see how we got into this mess and also how to get out of it. Classically a solution requires appeal to Newtonian ontology and laws. Quantum mechanically a solution needs the ontology and laws found in some quantum interpretations.

2 The Puzzle of Time Reversal in Quantum Theory

The conventional wisdom is that quantum theory – if we focus only on unitary evolution and ignore controversial features such as collapses – is time reversal invariant. In the Schrödinger representation, the quantum state is described by a wavefunction. If we focus on one particle $\psi = \psi(x, t)$, the evolution of the state ψ through time is given by the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{\mathcal{H}}\psi \tag{1}$$

where we'll assume throughout that we're working with a Hamiltonian $H = {p^2}/{2m} + V(x)$ that is time reversal invariant. (1) is a first-order parabolic partial differential equation. As such, it is of the same family as the heat equation

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u$$

where u may represent the temperature and κ is a real-valued coefficient. The heat equation, of course, is the poster child of temporally irreversible equations. Given their similar form, how can it be that they differ in this symmetry?

To be a bit more precise, normally one associates time reversal in classical spacetimes with the physical operation corresponding to a temporal reflection about a spatial axis: the past is swapped for the future. Then, in natural coordinates over the spacetime $\{x, y, z, t\}$, a passive interpretation of time reversal corresponds to flipping the coordinates $t' = -t$ and using $\{x', y', z', t'\}$. Put actively, one sticks with the same coordinates but changes the world – reverses the sequence of physical states — so that the description of the world exactly matches the description in the passive transformation. Call this "no frills" time reversal. Assuming time translation invariance, one flips the temporal sequence of states and that's it. If any other variable changes under the transformation (e.g., classical velocity) it happens "by logic and definition alone" from that flip.

Applying this operation to the Schrödinger equation, one immediately sees that it is not time reversal invariant in this sense. A minus sign pops out due to it being first-order in time. Typically, if $\psi(x, t)$ is a solution of (1) then $\psi(x, -t)$ is not. A symmetry should take solutions to solutions and non-solutions to non-solutions. Solutions to the Schrödinger equation are not solutions to the time-reversed Schrödinger equation, so the symmetry seems broken. That is precisely what we say about the heat equation when we explain that it is not time reversal invariant.

Readers familiar with the topic will demand that we stop right here. They will point out that Wigner 1931 taught us that time reversal in quantum theory is represented by *two* operations, a temporal reflection and complex conjugation $\psi \to \psi^*$. And it is certainly true that (1) is invariant under these combined operations. That is, if $\psi(x, t)$ is a solution of (1) then so is $\psi^*(x, -t)$, and vice versa. Quantum mechanics does indeed have this symmetry.

But is that the end of the issue? No. One can *call* the Wigner transformation "time reversal" if one wants and say that vindicates the conventional wisdom, yet that obscures the fact that the two symmetries are different. Dub them whatever you like, the operation of flipping temporal sequence is different than that plus implementation of complex conjugation. Or so it seems. If someone described the operation $L : \kappa \rightarrow -\kappa$ and then said that the heat equation was time reversal invariant because its solutions are invariant under the combined operation of time reversal and L, one would feel that was a bit of a cheat–or at least a kind of pun on the phrase "time reversal". Why in the world should κ flip sign due to reversing the temporal sequence? It doesn't follow from logic and definition alone. Same in quantum theory. Why should $K : i \rightarrow -i$ when reversing the temporal order? i is a number without spatiotemporal dimensions, so why should it flip?

It is no wonder then that there has been a fair number of people in the philosophical foundations of physics puzzling over time reversal in quantum theory (e.g., Albert 2000, Allori 2022, Callender 2000; Earman 2002; Gao 2022; Lopez 2021; Malament 2004; Roberts 2017, 2022; Struyve 2022). Different camps emerged differing over whether the Wigner symmetry should count as time reversal symmetry. One group holds that quantum theory is not invariant under time reversal, strictly speaking, but it is invariant under what we might call "motion reversal" (implemented via Wigner's symmetry). Given the interpretive complications with quantum theory, it wouldn't be surprising if two similar symmetries that overlap classically were conflated quantum mechanically. The other camp holds that Wigner symmetry is or warrants the label time reversal invariance once this symmetry is properly understood.

Lately, however, there have been efforts at a kind of rapprochement between the two camps. Callender 2023 and Roberts 2022 hope to show that Wigner symmetry does follow from no frills time reversal. The present argument is in this spirit. Success in this endeavor would be the best possible resolution. After all, one has a strong suspicion that there is still more to the story, that disambiguation (or not) is too simple. As Schrödinger 1931 remarks when comparing (1) with the Fokker-Planck equation (a parabolic equation like (2)), "whilst in both cases the differential equation is of first order in time, the presence of a factor $\sqrt{-1}$ confers to the wave equation a hyperbolic or, physically stated, reversible character at variance with the parabolic-irreversible character of the Fokker[-Planck] equation" (Chetrite 2021, section 2.5). So the similarity with the heat equation may be superficial. Due to the i, (1) "feels" more like a hyperbolic equation than a parabolic one, which, after all, is why it's called a "wave equation" despite its parabolic nature. A more satisfying explanation would offer a deeper insight into the connection between complex conjugation and time reversal.

In what follows I provide such an explanation in three steps. The first step is to see that this puzzle can arise even in purely classical physics. The second identifies what it takes to extricate ourselves from the classical puzzle. The third simply takes the classical solution's counterpart in the quantum context. No doubt this last step will be controversial because it involves an interpretation of quantum theory. But if I'm right, the puzzle over time reversal in quantum theory arises because standard quantum mechanics gives us only "half" a theory.

3 The Classical Schrödinger Equation

In this section we'll stay entirely within the classical mechanical regime but shift to a representation that looks very quantum. We'll derive the so-called "classical Schrödinger equation" of Schiller 1962, Rosen 1964, and others. This equation arises from treating classical mechanics as a kind of field theory. The field theory is itself motivated from classical statistical mechanics. We'll use the Hamilton-Jacobi formulation of classical mechanics. Although it can describe a single particle, the Hamilton-Jacobi picture is naturally suggestive of a continuous ensemble of particles evolving in configuration space beginning with different initial conditions. And from this theory a field picture is natural, one with wavefronts evolving in configuration space. This picture is the reason why Hamilton-Jacobi theory is usually taught as a precursor to quantum theory. To get the classical Schrodinger equation we'll simply bundle these waves into a Schrodinger-like form.

We will consider particles of mass m evolving in in a potential V in a Cartesian system $x = (x, y, z)$. We'll denote with q_i and p_i the generalized coordinates, where $i = 1, ..., n$ and n is the number of degrees of freedom of the system. The Lagrangian is

$$L(x, \dot{x}, t) = \frac{1}{\sqrt{2}} m\dot{x}^2 - V(x, t)$$

and the action is

$$S = \int_{t_I q_I}^{t_F q_F} L(x, \dot{x}, t)$$

where I is the initial time and location and F is the final time and location. Then using identities found in any good analytical mechanics textbook (e.g., Johns 2016), one derives the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x, t) = 0 \tag{2}$$

The surfaces of constant S are regarded as wavefronts. These are sets of points q at time t for which $S(q, t)$ =constant. (2) describes the evolution of these wavefronts with time. While Hamilton-Jacobi wavefronts are often derived from knowledge of particle trajectories, it's important to recognize that this is not necessary. We can simply begin with (2) and regard it as an independent equation governing these wavefronts.

To extract individual trajectories from (2), one can associate particle paths with the characteristics of (2). Jacobi's Theorem tells us that $p_i = \frac{dS}{dx}$, and we can use this relation and a separation of variables to deduce the equation of motion for particular initial positions $x = (x(t, x_0))$. In this sense Hamilton-Jacobi theory encodes a kind of "particle-wave" duality, which is what attracted Schrödinger to it when he discovered his famous equation. One can begin with waves and extract particle trajectories or vice versa.

The Hamilton-Jacobi theory is naturally suggestive of an ensemble picture. The S-function gives us momenta via $p = \nabla S$. Yet via the classical Lagrangian and its conjugate momentum we know that $p = m\dot{x}$. So $m\dot{x} = \nabla S$. The S-function therefore describes a velocity field all over configuration space. In the single-particle case, one particle "surfed" the "wave" described by S. But we can easily imagine an infinite ensemble of particles differing in their initial positions $x = (x(t, x_0))$ surfing the same S-function but tracing out their respective trajectories.

Let this ensemble be distributed with density $R^2(x, 0) \geq 0$. As the particles evolve according to (2), we can watch this distribution $R^2(x, t)$ evolve in time. Since the particles move without sources or sinks, they must obey a continuity equation

$$\frac{\partial R(x,t)^2}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{m}\frac{\partial S(x,t)}{\partial x}R^2(x,t)) = 0 \tag{3}$$

This is a conservation law for probability, one implied by the classical Liouville theorem. Writing the Hamilton-Jacobi equation for an ensemble as the combined equations (2) and (3) describe a statistical mechanics of classical particles. Solutions are obtained by specifying the initial S and R, $S(x, 0) = S_0(x)$ and $R(x, 0) = R_0(x)$. $S_0(x)$ determines an initial momentum field that shapes the development of the initial distribution. We arrive at a kind of field theory describing two partially coupled fields, $S(x, t)$ and $R^2(x, t)$.

Readers familiar with the Madelung decomposition of the Schrödinger equation know that if we write the wavefunction in polar form and separate real and imaginary parts we obtain two real equations, a "quantum" Hamilton-Jacobi equation and a continuity equation (Holland 1993). Since we have a Hamilton-Jacobi equation and continuity equation in equations (2) and (3), can we perform a "reverse Madelung decomposition" and combine these equations to form a "classical" Schrödinger equation.

To get the classical Schrödinger equation, we simply package this familiar physics into a single compact equation. Introduce complex classical wavefunctions by joining together the action S and the above R to form:

$$\psi_{CL} = R(x,t)exp(iS(x,t)/\hbar) \tag{4}$$

where $\hbar$ is added merely to mimic the appearance of the quantum.[1] Combined together into one complex-valued equation, (2) and (3) are equivalent to

$$i\hbar\frac{\partial\psi_{CL}(x,t)}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t) - Q(x,t))\psi_{CL}(x,t) \tag{5}$$

which is the so-called *classical Schrödinger equation* (Schiller 1962; Rosen 1964). Here U is the classical potential and we assume it time-independent $U(x, t) = U(x)$. Q is what is sometimes dubbed the quantum potential. It has the form

$$Q = -\frac{\hbar^2}{2m}\frac{1}{R}\frac{\partial^2 R}{\partial x^2}.$$

As we know from Bohmian mechanics (Bohm 1952), Q carries all the "quantum" aspects of nature. Intuitively, (5) is the Schrödinger equation (1) with its quantum aspects "subtracted" out.

[1] It will drop out when equations of motion are derived.

This "classical Schrödinger equation" should not be confused with the "classical Schrödinger equation" of Koopman 1931, von Neumann 1932, and Sudarshan 1976. The present equation is nonlinear and its wavefunction's domain is configuration space, whereas the Koopman-von Neumann-Sudarshan equation is linear and its wavefunction's domain is phase space.[2] The nonlinearity means that superpositions of solutions are not generally also solutions. If ψ_1 is a solution and ψ_2 is a solution, then $\psi_1 + \psi_2$ is not generally a solution; the exceptions are when ψ_1 and ψ_2 have no common support or when one is a multiple of the other. Interference patterns also do not develop, as one would expect.[3]

4 Time Reversal of the Classical Schrödinger Equation

Consider time-reversal in this dynamics. Because each trajectory in the ensemble is governed by Newtonian dynamics, we expect the evolution to be time-reversal invariant in a no frills sense. And it is. We're just running a bunch of Newtonian particles forward and backward in time. Newton's second law is time reversal invariant, so the ensemble should be too.

What's interesting and a bit surprising is that in this formalism one achieves this temporally reversed evolution by *complex conjugation of (5).* We're in exactly the same situation with the classical Schrödinger equation as we were with the quantum Schrödinger equation! Just as with the Schrödinger equation, temporal reflection will bring a minus sign out in front of (5). The equation is non-linear but still a first-order parabolic equation. Solutions to (5) will not be solutions to this equation:

$$-i\hbar\frac{\partial\psi_{CL}(x,t)}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t) - Q(x,t))\psi_{CL}(x,t) \tag{6}$$

However, if we take the complex conjugate when we time reverse, then solutions of (5) will be mapped one-to-one to solutions of

$$i\hbar\frac{\partial\psi^*_{CL}(x,t)}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t) - Q(x,t))\psi^*_{CL}(x,t) \tag{7}$$

which is the complex conjugate of (5).

We're in precisely the same situation as we were quantum mechanically. But the beauty of this case is that we know exactly what is going on because we have a good theoretical understanding of classical mechanics, its ontology, laws, and the derivation of the wave equation (5). This understanding will help us extricate ourselves from the quantum puzzle.

To drive the point home, let's be clear. Forget the derivation of (5). Suppose you were just given (5). This is a perfectly conceivable world. You've got your waves and from these you obtain your probability distributions. To get the distributions, you supplement (5) with a classical version of Born's rule. Perhaps you attach a Copenhagen-like

[2]For a philosophical lesson we learn from Koopman-von Neumann-Sudarshan, see Callender 2024.

[3]For further discussion of (5), see Benseny, Tena, and Oriols 2016, Holland 1994, Ghose 2018, Rosen 1964, 1965, Schiller 1962, 1986, Schleich et al 2013.

interpretation to this, or something else. It is a theory along the line suggested by Prigogine 1996 and Krylov 1979 and developed philosophically by McCoy 2020. You have an empirically adequate (to the classical world) theory in a classical statistical mechanical universe. You get along perfectly fine. You don't have or believe that you need a trajectory-based classical physics.

Still, you're a bit puzzled by time reversal. (5) is a first-order in time equation like the quantum Schrödinger equation. On a "no frills" sense of time reversal, it is simply not time reversal invariant. But you notice that if you take the complex conjugate of the wavefunction when time reversing, then solutions in one temporal direction map to solutions in the other. What reason would you have to take the complex conjugate when time reversing? As before, complex conjugation doesn't follow from temporal reflection by logic or definition alone. We could readily envision classical mechanical counterparts of all the concerns raised in the introduction.

If we assume that really the theory is time reversal invariant – and we should because we know what's going on, to which we will turn in a moment – the problem is that we need the classical momenta to reverse under time reversal. In the other classical representations, we know that (q, p) going to $(q, -p)$ will cause the forward in time trajectories to retrace their paths in the reverse time direction. But in this representation – where we only have (5) – we lack reason to flip the momenta when we implement a temporal reflection. From Jacobi's theorem we have

$$p = \frac{\partial S}{\partial q} \tag{8}$$

but this is *spatial variation,* not temporal variation. If we're strict about only reversing what follows from a simple temporal reflection, then we have no reason to turn the momenta around. To get the classical waves to turn around we need the classical phase to flip. Complex conjugation achieves this as it reverses $iS/\hbar$ in (4), which in turn flips the momenta. With what we have, however, this isn't motivated from a no frills temporal reflection.

5 The Classical Solution

There is no mystery in this case about what's going on. We know more than (5). In a classical world there are more than just waves rippling through high dimensional configuration space. The wave description was built up from a statistical mechanics of individual particles and these particles ultimately are governed by Lagrangian or Hamiltonian or Newtonian mechanics. These resources associated with the particle ontology give us the ability to link the above momenta with temporal variation.

Suppose we have a classical free particle. It is governed by (2). Its solutions will be spherical wave fronts, and we'll know that the momentum p is given by equation (8) above. But we also know a whole lot more if we assume that we have particles with trajectories. Importantly, we also know that the Lagrangian is $L(x, \dot{x}, t) = (1/2)m\dot{x}^2 - V(x, t)$. The particle's canonical momentum is $p_i = \partial L/\partial \dot{q}$. With our Lagrangian, that means $p = m\dot{x}$.

So $m\dot{x} = m^{dx}/_{dt} = \nabla S$. See Holland 1993, 51. Alternatively, with a three-dimensional delta function we can extract the motion of a single particle directly from a solution S of (2) (see Rosen 1986, 690). This point is absolutely crucial. The canonical momentum on its own in our "wave picture" isn't necessarily tied to temporal variation. But when we bring in the resources and assumptions of a particle ontology traversing well-defined trajectories, we connect this to velocities. That connects the *temporal variation* of the ontology with the *spatial variation* of the phase. The classical Schrödinger representation does not give us this (or if you prefer, my imagined above world where we're simply given (5) doesn't). Only when we look "under the hood" of this representation do we see this connection. By "under the hood" I mean classical particle mechanics. [4]

And of course we know that classical mechanics is time reversal invariant in a no frills sense. Again sticking with our above single particle Lagrangian, we can use the second order Euler-Lagrange equations to derive Newton's equation of motion:

$$mx = -\ddot{\nabla} V|_{x=x(t)}$$

which is time reversal invariant in a no frills sense.

In the classical Schrodinger equation representation, i does not go to $-i$ under time reversal. Or it shouldn't if one uses the full resources attached to the ontology. Really i never flips. It only looks like it flips if one ignores some of classical mechanics. What really happens is that time "flips", which in turn "flips" velocity, which in turn "flips" the classical phase; when that reflected phase is then used to compose a classical wavefunction, that wavefunction is the complex conjugate of the original classical wavefunction. The complex conjugation is simply a downstream consequence of reflecting time.

Opening the hood shows us that complex conjugation does indeed follow by logic and definition alone from a simple no frills temporal reflection. The "problem" of time reversal in the Schrödinger representation of classical physics is therefore solved.

6 Back to Quantum Time Reversal

In standard quantum mechanics we are in the situation where we only have the quantum counterpart of (5). It's therefore hard to see a good reason to take the complex conjugate when time reversing a system. Complex conjugation does not follow from $t \to -t$ if we restrict ourselves only to the Schrödinger equation. Hence time reversal is very puzzling.

What makes this hard to tolerate is that quantum theory *almost* has the resources to extract itself as we did above. After all, a Madelung decomposition of the Schrödinger equation will yield two real equations, a quantum Hamilton-Jacobi equation and a continuity equation. Using a hydrodynamic analogy, we want to say that the probability current j defines a velocity for us via

[4] One could also point out that classical hydrodynamics is an an isomorphic situation. Like our systems here, the fluid is described by a continuity equation and an Euler equation (a field-theoretic expression of Newton's second law). With the usual restrictions, classical hydrodynamics is considered time reversal invariant. The reason is that the current (density times velocity) reverses sign due to the velocity flipping. Without "opening the hood" on the physics of the fluid, we might not see that the theory is time reversal invariant.

$$v = \frac{j}{|\psi(x,t)|^2} \tag{9}$$

or using the phase of the quantum Hamilton-Jacobi equation via

$$v = \nabla S$$

.

We're prohibited from doing so because – we're told – we all know that quantum mechanics is incompatible with always-determinate trajectories, and hence there can be no velocities. This conventional wisdom is repeated again and again throughout quantum theory. In the Sakurai 1994 textbook the author uses scare quotes around the equal sign in equation (N), and in Shimbori and Kobayashi 2000 the authors use scare quotes around the word velocity. Sakurai uses scare quotes because he thinks always-determinate trajectories and velocities will contradict the quantum uncertainty principle[5]; see Romano 2021 for a convincing reply. Shimbori and Kobayashi 2000 don't say why they use scare quotes but presumably it is due to this conventional wisdom. In the philosophy literature, Earman 2002 and Gao 2022 recognize that the quantum phase must turn around, but neither explicitly connect this via logic and definition alone to a temporal reflection. Gao 2022 suggests that assuming the continuity equation invariant under time reversal is a requirement of meaning, and hence due to definition. However, in the next paragraph we see that the concept of the best explanation is really doing the work. Physics in the time-reversed world would be unexplained, he says, if we do not reverse the momentum–but that is just what happens in time reversed worlds in a non-time reversal invariant theory. Earman 2002 writes:

> How can the information about the direction of motion of the wave packet be encoded in $(x, 0)$? Well (when you think about it) the information has to reside in the phase relations of the components of the superposition that make up the wave packet. And from this it follows that the time reversal operation must change the phase relations.

As congenial as what Earman says is to the present perspective, "when you think about it" and "has to" are not derivations. The spatial variation in the phase is only definitionally linked to the temporal variation in a temporal reflection when one removes the above scare quotes.

Remove the square quotes and we remove the problem. What helped us understand complex conjugation's connection to time reversal in the case of the classical Schrödinger equation was that we already knew or posited the ontology and individual particle dynamics. We opened the hood and looked at the engine. What hinders us in the quantum case is that we don't know what's going on. The infamous measurement problem leaves us not knowing what the ontology or extra dynamics (if any) are.

[5] Sakurai: "we would like to caution the reader against a too literal interpretation of j as ρ times the velocity defined at every point in space, because a simultaneous precision measurement of position and velocity would necessarily violate the uncertainty principle" (1994, 102-103).

However, there are interpretations of quantum mechanics that are especially well-suited for tackling this question. One is Bohmian mechanics. Bohmian mechanics can be derived via a Madelung decomposition of the Schrödinger equation to a "quantum" Hamilton-Jacobi equation and continuity equation – essentially the reverse of how we derived the "classical" Schrödinger equation. Non-relativistic Bohmian mechanics also posits particles guided by a dynamical law, like Newtonian mechanics, and in many ways quantum mechanics is to Bohmian mechanics as statistical mechanics is to classical Newtonian dynamics. This similarity raises the possibility that we can understand complex conjugation's connection to time reversal in the same way as we did classically. Perhaps we can open the hood and see a rationale for complex conjugation when time reversing by studying the Bohmian engine.

Indeed, we can. Bohmian mechanics describes particles with always-determinate positions, X. The dynamics for these particles come in many equivalent forms. For our needs the "Newtonian" second-order formulation may be the most transparent. This formulation was the original one presented by Bohm. In this representation, the Bohmian dynamics for the particles are given by Newton's second law, except now the above quantum potential joins the classical potential:

$$m\frac{d^2X(t)}{dt^2} = -\nabla(Q+V) \tag{10}$$

where we assume that all the particles have the same mass and that $V(x,t) = V(x)$. Because the quantum potential Q does not flip sign under time reversal, we can see that the Bohmian dynamics is time reversal invariant in a completely simple "no frills" sense. This theory is now written in terms of two real-valued functions, R, and S, just as classical Hamilton-Jacobi theory is. The dynamics requires that particle velocities satisfy $p = mv = \nabla S$, as in classical Hamilton-Jacobi theory. Here I stress that there is an "equals sign" without scare quotes.

We can now transform into the complex-valued wave representation. When we do, we see that in order to get back the time-reversed (in the no frills sense) Bohm trajectories, we need to take the complex conjugate of the wavefunction. Without the underlying ontology, that transformation seems to have no rationale. But when we look under the hood at the assumed Bohmian ontology and laws, we see that complex conjugation does follow from $t \to -t$. As in the classical setting, this no frills temporal reflection alters the sign of the velocities of the particles. And via $p = mv = \nabla S$, that changes the sign of the phase. That then demands complex conjugation.[6]

[6]Note that I must assume the so-called "nomological" interpretation of Bohm's theory for this to work (see Dürr, Goldstein, and Nino Zanghì 1997; Callender 2015). On this picture, the particles are understood as the primitive ontology and the wavefunction is understood as nomological, like the Hamiltonian. If the wavefunction is on par with the particles, we end up with the original problem restated in the Bohm formalism. That is, perhaps the velocities flip as suggested in the text but the phase does not; that would indicate that the guidance equation (equation (9) above) doesn't hold under time reversal. That possibility exists because the wavefunction is its own entity on non-nomological views. On nomological interpretations, by contrast, the particles (or fields, or whatever beables are posited) are the "boss", and the wavefunction is part of what governs or best describes the particle motions. If velocity flips, then so does the phase. This restriction to the nomological

Essentially the same reasoning holds also in the "Newtonian Quantum Mechanics" interpretation of Hall et al 2014 and Sebens 2015. According to this theory, there are many classical worlds evolving (and interacting) via a Newtonian force law that is manifestly time reversal invariant in a no frills sense. The time reversal invariance of the fundamental ontology forces the phase to change sign. And when we compose the wavefunction from the more basic entities in the theory, complex conjugation occurs when time reversing, just as it does here when composing wavefunctions in both the classical and quantum Hamilton-Jacobi representations. Sebens claims that explaining time reversal is an advantage of his interpretation. I agree that it is an advantage of any interpretation that it link the spatial variation of phase information with time.

7 Conclusion

Time reversal in quantum theory has long been puzzling. The main point of this paper is that precisely this same puzzle arises also in classical physics, and in particular, in the representation of classical physics via the classical Schrödinger equation of Schiller and Rosen. I think many readers will be surprised to learn this. What does it mean? To me, it teaches us that the puzzle of time reversal in quantum theory originates from using a theory stripped of its ontology. Once the ontology and its dynamics are added, all becomes clear. Presented in its real-valued form with a full Bohmian ontology and laws (or other interpretation with similiar effect), the theory is time reversal invariant in a no frills sense. We don't need to take the complex conjugate of anything. Translated into a complex-valued representation, complex conjugation is simply a consequence of $t \to -t$. Normally, we cannot see this because we are working with only "half" the theory. But if we imagine that a good interpretation provides the other half of the story, we recognize the engine that derives complex conjugation from temporal reflections.[7]

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view may be seen as a limitation of my proposal; alternatively, to the extent that it nicely resolves the puzzle of time reversal, it may be received as an additional reason in favor of the nomological picture of the Bohmian wavefunction.

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